

## Generalized scaling function at strong coupling

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**Nikolay Gromov**

*Service de Physique Théorique, CNRS-URA 2306 C.E.A.-Saclay,  
91191 Gif-sur-Yvette, France, and  
Laboratoire de Physique Théorique de l'Ecole Normale Supérieure et  
l'Université Paris-VI,  
Paris, 75231, France, and  
Theory department, St.Petersburg INP,  
Gatchina, 188 300, St.Petersburg, Russia  
E-mail: nikgromov@gmail.com*

**ABSTRACT:** We considered folded spinning string in  $AdS_5 \times S^5$  background dual to the  $\text{Tr}(D^S \Phi^J)$  operators of  $\mathcal{N} = 4$  SYM theory. In the limit  $S, J \rightarrow \infty$  and  $\ell = \frac{\pi J}{\sqrt{\lambda} \log S}$  fixed we compute the string energy with the 2-loop accuracy in the worldsheet coupling  $\sqrt{\lambda}$  from the asymptotical Bethe ansatz. In the limit  $\ell \rightarrow 0$  the result is finite due to the massive cancelations with terms coming from the conjectured dressing phase. We also managed to compute all leading logarithm terms  $\frac{\ell^{2m} \log^n \ell}{\lambda^{n/2}}$  to an arbitrary order in perturbation theory. In particular for  $m = 1$  we reproduced results of Alday and Maldacena computed from a sigma model. The method developed in this paper could be used for a systematic expansion in  $1/\sqrt{\lambda}$  and also at weak coupling.

**KEYWORDS:** Supersymmetry and Duality, Duality in Gauge Field Theories, AdS-CFT Correspondence, Integrable Field Theories.

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## 1. Introduction

In this paper we will consider the  $sl(2)$  sub-sector of the AdS/CFT duality describing the operators of the form  $\text{Tr}(D^S \Phi^J)$ . This sector is known to be closed perturbatively to all orders in the gauge coupling. This means that the operators with  $S$  derivatives and  $J$  scalar fields mix only with each other under renormalization. The corresponding mixing matrix in the planar 't Hooft limit is believed to be an integrable Hamiltonian of an  $sl(2)$  spin chain for all values of the 't Hooft coupling  $\lambda$ . This assumption drastically simplifies computation of anomalous dimensions of these operators which could be done by mean of

a Bethe ansatz, based on the S-matrix approach [1]. In the  $sl(2)$  subsector the asymptotic all-loop Bethe equations read [2–5]

$$\left(\frac{x_k^+}{x_k^-}\right)^J = \prod_{j \neq k}^S \left(\frac{x_k^+ - x_j^-}{x_k^- - x_j^+}\right)^{-1} \frac{1 - 1/(x_k^+ x_j^-)}{1 - 1/(x_k^- x_j^+)} \sigma^2(u_k, u_j), \quad (1.1)$$

where  $x_k^\pm \equiv 2\pi \frac{u_k + i/2}{\sqrt{\lambda}} + \sqrt{4\pi^2 \left(\frac{u_k + i/2}{\sqrt{\lambda}}\right)^2 - 1}$  and  $\sigma^2$  is the famous dressing factor [6, 4]. If one solves this equation and finds set of  $u_k$ 's the anomalous dimension is given by

$$\gamma(\lambda, S, J) = \frac{\sqrt{\lambda}}{2\pi} \sum_{j=1}^S \left(\frac{i}{x_k^+} - \frac{i}{x_k^-}\right). \quad (1.2)$$

At the string side of the duality the corresponding state is a folded string living in  $AdS_3 \times S^1$  and carrying large angular momenta  $S$  and  $J$ . The energy of the string is given by  $S + J + \gamma(\lambda, S, J)$  via the AdS/CFT duality [7] and the world-sheet sigma model coupling is  $\lambda^{-1/2}$ .

The equations (1.1) are still rather complicated. To simplify the problem we will consider the limit introduced in [8, 9] when  $J, S \rightarrow \infty$  and

$$\ell = \frac{\pi J}{\sqrt{\lambda} \log S} \quad (1.3)$$

is fixed. In this limit the anomalous dimensions scales as  $\log S$  [10] and one defines the so-called generalized scaling function  $f(\lambda, \ell)$  by

$$\Delta - S - J = \gamma = \lambda^{1/2} \frac{f(\lambda, \ell)}{\pi} \log S \quad (1.4)$$

or equivalently

$$f(\lambda, \ell) = \frac{\gamma(\lambda, \ell, J)\ell}{J}. \quad (1.5)$$

We will compute this quantity as an expansion in  $1/\sqrt{\lambda}$  keeping a full functional dependence on  $\ell$ .

$$f(\lambda, \ell) = f_{cl}(\ell) + \lambda^{-1/2} f_{1-loop}(\ell) + \lambda^{-1} f_{2-loop}(\ell) + \dots \quad (1.6)$$

This object was studied intensively at both strong and weak coupling [3, 4, 8, 9, 11–23]. The strong coupling expansion is known up to two loops to be

$$f_{cl}(\ell) = \sqrt{\ell^2 + 1} - \ell \quad (1.7)$$

$$f_{1-loop}(\ell) = \frac{\sqrt{\ell^2 + 1} - 1 + 2(\ell^2 + 1) \log\left(1 + \frac{1}{\ell^2}\right) - (\ell^2 + 2) \log \frac{\sqrt{\ell^2 + 2}}{\sqrt{\ell^2 + 1} - 1}}{\sqrt{\ell^2 + 1}} \quad (1.8)$$

$$f_{2-loop}(\ell) = -C + \ell^2 (8 \log^2 \ell - 6 \log \ell + q_{02}) + \mathcal{O}(\ell^4), \quad (1.9)$$

where  $C$  is Catalan's constant and  $q_{02}$  is some number. The two-loop term (1.9) have not been yet computed for an arbitrary  $\ell$ . Only a couple of terms in small  $\ell$  expansion are

known [22]. In this paper we will compute  $f_{2\text{-loop}}(\ell)$  directly from Bethe ansatz (1.1). We will see that the result is finite in  $\ell \rightarrow 0$  limit only due to massive cancelations with terms coming from the dressing factor.

Our method is similar to [17], where the one loop result (1.8) of [9] was confirmed from the Bethe ansatz (1.1). We will expand (1.1) first in the classical limit  $S \sim J \sim \sqrt{\lambda}$  [24] and then pass to the limit described above. This order of limits is exactly the same as in perturbative expansion of the worldsheet sigma model [21] and we are free from the potential order-of-limits problem.

It is known that a two-loop computation in Bethe ansatz is qualitatively more complicated problem than a one-loop computation. At two loops the discrete behavior of the Bethe roots  $u_k$  becomes important [25]. In this paper we will show how to efficiently override these difficulties and rewrite (1.1) as a quadratic equation.

Basing on some natural assumptions about the behavior of the strong coupling expansion at small  $\ell$  we managed to compute all the terms of the form  $\frac{\ell^{2m} \log^n \ell}{\lambda^{n/2}}$  in  $f(\lambda, \ell)$  using just 1-loop result for  $f(\lambda, \ell)$ . In a particular case  $m = 1$  we found a perfect agreement with [19].

The paper is organized as follows: in section 2 we expand the Bethe equations in classical limit and rewrite it as a simple quadratic equation, in section 3 we focus on the terms coming from the Hernandez-Lopez phase and “anomaly” contribution, in section 4 we combine all the contributions together and write down our 2-loop correction to the scaling function, in section 5 we subtract leading logarithms at all orders in  $1/\sqrt{\lambda}$ , in section 6 we conclude. Appendix A contains some intermediate computation, in appendix B we write an expansion in powers of  $\ell$  and in tables 1, 2 and 3 we give our results in *Mathematica* syntax.

## 2. Strong coupling expansion of Bethe equations

In this section we will expand Bethe equations (1.1) in the strong coupling limit  $\lambda \rightarrow \infty$ . We will also keep  $S, J \sim \sqrt{\lambda}$ . It is well known that in these settings the Bethe roots  $u_k$  scale like  $\sqrt{\lambda}$  [24]. It is convenient to introduce

$$x_k \equiv 2\pi \frac{u_k}{\sqrt{\lambda}} + \sqrt{4\pi^2 \left( \frac{u_k}{\sqrt{\lambda}} \right)^2 - 1} \tag{2.1}$$

so that  $x_k \sim 1$ . Then  $x_k^\pm$ , which enter the Bethe equations (1.1) and the expression for anomalous dimensions (1.2), can be expanded in  $1/\sqrt{\lambda}$

$$x_k^\pm = x_k \pm \frac{i\alpha(x_k)}{2} + \frac{\alpha^2(x_k)}{4x_k(x_k^2 - 1)} \pm \dots \tag{2.2}$$

where  $\alpha(x) = \frac{4\pi}{\sqrt{\lambda}} \frac{x^2}{x^2 - 1}$ . It will be very useful to introduce a resolvent

$$\mathcal{G}(x) = \frac{1}{J} \sum_j \frac{1}{x - x_j} . \tag{2.3}$$

We will also use  $g = \frac{\sqrt{\lambda}}{4\pi}$  for convenience.

Now we can express in a compact form the expansion of anomalous dimension (1.2). In the notations introduced above for symmetric distribution of roots it reads

$$\frac{\gamma(g)}{J} = - \left( 2\mathcal{G} + \frac{3\mathcal{G} - 3\mathcal{G}' - 21\mathcal{G}'' - 10\mathcal{G}^{(3)} - \mathcal{G}^{(4)}}{384g^2} + \mathcal{O}\left(\frac{1}{g^4}\right) \right) \Bigg|_{x=1}. \quad (2.4)$$

To expand Bethe equations one usually takes log of both sides first. To fix the branch of the logarithm one should add  $2\pi i n_k$  where  $n_k$  are some integer numbers called mode numbers [24]. The expansion is then straightforward and leads to

$$-\frac{2\pi n_k}{J\alpha(x_k)} = \frac{2}{J} \sum_{j \neq k} \frac{1}{x_k - x_j} + \frac{\gamma(g) + J}{Jx_k} + \frac{x_k}{4g^2} \left( \frac{x_k^4 + 4x_k^2 + 1}{(x_k^2 - 1)^4} - \frac{\mathcal{G}(1) + 3\mathcal{G}'(1) + \mathcal{G}''(1)}{3(x_k^2 - 1)^2} \right) + \frac{\mathcal{V}_{\text{HL}}(x_k)}{J\alpha(x_k)} + \frac{\pi\rho'(x_k)}{J} \left( \coth(\pi\rho) - \frac{1}{\pi\rho} \right) + \mathcal{O}\left(\frac{1}{g^3}\right). \quad (2.5)$$

Let us explain the origin of the different terms. The first line comes from the Bethe equation with the full dressing phase, except the Hernandez-Lopez phase [26, 27] which results in the first term in the second line. The second term in the second line is known under the name of ‘‘anomaly’’ and comes from the terms in the product with  $j - k \sim 1$  [28]. In this expansion we noticed that the terms  $\mathcal{G}^{(n)}(1/x_k)$  appearing all the way cancels out when the 2-loop dressing phase is taken into account. This cancelation could be a very restrictive condition on the phase and is probably equivalent to the crossing.<sup>1</sup>

Let us emphasize once more that the 2-loop dressing phase is taken into account, but its contribution is not explicitly seen in (2.5). The resulting equation is much simpler and does not contain  $\mathcal{G}^{(n)}(1/x_k)$  terms when we mix expansion of the Bethe equation without dressing phase with 2-loop dressing phase.

In the paper [27] a very compact representation of the Hernandez-Lopez phase [26] was derived which we will use here

$$\frac{\mathcal{V}_{\text{HL}}(x)}{\alpha(x)} = \int_{-1}^1 \left( \frac{1}{x-y} + \frac{1}{x} + \frac{1}{1/y-x} \right) \partial_y \left( \frac{\mathcal{G}(1/y) + y^2\mathcal{G}(y) - 2y\mathcal{G}(1)}{g(y^2-1)} \right) \frac{dy}{2\pi}, \quad (2.6)$$

where the integration goes along the upper half of the unit circle  $|x| = 1$ .

The anomaly term (the last term in the second line of (2.5)) contains density  $\rho$  of the roots  $u_k$ . We will use two different densities

$$\rho \equiv \frac{1}{\partial u_k / \partial k}, \quad \varrho \equiv \frac{1}{J \partial x_k / \partial k}, \quad (2.7)$$

which are trivially related

$$\rho(x) = J\alpha(x)\varrho(x), \quad \varrho(x) = -\frac{\mathcal{G}(x+i0) - \mathcal{G}(x-i0)}{2\pi i}, \quad (2.8)$$

where  $\alpha(x) = \frac{x^2}{g(x^2-1)}$ .

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<sup>1</sup>this cancelation appears also at higher orders. We thank to P.Vieira for discussing this point.

To proceed one have to specify a particular set of mode numbers  $\{n_k\}$ . Different sets of mode numbers will lead to different solutions of the Bethe ansatz. They correspond to different string motions. The one corresponding to the simplest folded string is

$$n_k = -1 \quad , \quad k = 1, \dots, S/2 \quad ; \quad n_k = +1 \quad , \quad k = S/2 + 1, \dots, S \quad . \quad (2.9)$$

On the gauge theory side this choice corresponds to the twist  $J$  operators (i.e. operators with all Lorentz indices symmetrized and traceless). We see that this set of  $n_k$ 's respects  $x_k \rightarrow -x_{S-k}$  symmetry and the resulting distribution of roots should be symmetric with respect to the origin

$$\varrho(-x) = \varrho(x) \quad , \quad \mathcal{G}(-x) = -\mathcal{G}(x) \quad . \quad (2.10)$$

When  $S \rightarrow \infty$  the roots are distributed on two symmetric cuts  $\mathcal{C} = (-b, -a) \cup (a, b)$  with  $a \sim 1$  and  $b \sim S/\sqrt{\lambda}$  [17]. It is important that the upper limit of the distribution scales like  $S/\sqrt{\lambda}$ . We will also see that the resolvent we introduced scales like 1 in our limit

$$\mathcal{G}(x), \varrho(x) \sim 1 \quad \text{for } x \sim 1 \quad . \quad (2.11)$$

## 2.1 Quadratic equation

Now we are coming to an important step in our calculation. We will rewrite (2.5) as a quadratic equation. To convert (2.5) into a quadratic equation we are using the standard trick - we multiply the equation by  $\frac{1}{J(x-x_k)}$  and sum over  $k$ . Using that

$$\sum_{k \neq j} \frac{2}{J^2(x-x_k)(x_k-x_j)} = \mathcal{G}^2(x) + \frac{1}{J} \mathcal{G}'(x) \quad , \quad (2.12)$$

where the last term is irrelevant for us since it is suppressed by  $1/J$ . We arrive at

$$-\frac{c^2(x)}{4} = \mathcal{G}^2(x) + \frac{\gamma + J \mathcal{G}(x)}{J} + \frac{\mathcal{F}(x)}{\ell^2} \quad . \quad (2.13)$$

This is our main equation which we will use to compute  $f(\lambda, \ell)$ . We introduced  $\mathcal{F}(x) = \mathcal{F}_0(x) + \mathcal{F}_{\text{HL}}(x) + \mathcal{F}_{\text{An}}(x)$  with

$$\mathcal{F}_0(x) = \frac{\ell^2}{g^2} \int_{\mathcal{C}} \frac{\varrho(y)}{x-y} \frac{y}{4} \left( \frac{y^4 + 4y^2 + 1}{(y^2 - 1)^4} - \frac{\mathcal{G}(1) + 3\mathcal{G}'(1) + \mathcal{G}''(1)}{3(y^2 - 1)^2} \right) dy \quad (2.14)$$

$$\mathcal{F}_{\text{HL}}(x) = \frac{\ell^2}{g} \sum_k \frac{1}{J(x-x_k)} \frac{g \mathcal{V}_{\text{HL}}(x_k)}{J \alpha(x_k)} \quad (2.15)$$

$$\mathcal{F}_{\text{An}}(x) = \ell^2 \sum_k \frac{\pi \rho'(x_k)}{J^2(x-x_k)} \left( \coth(\pi \rho(x_k)) - \frac{1}{\pi \rho(x_k)} \right) \quad . \quad (2.16)$$

So far we did not get a closed algebraic equation on the resolvent  $\mathcal{G}$ . We introduced above a new object  $c(x)$  defined by

$$-c^2(x) = \sum_k \frac{8\pi n_k}{J^2 \alpha(x_k)(x-x_k)} = 16\pi \frac{g}{J} \int_a^b \frac{\varrho(y)}{x-y} \left( 1 - \frac{1}{y^2} \right) dy \quad , \quad (2.17)$$

which depends on the resolvent. We see that (2.13) is some complicated nonlinear integral equation. Notice that  $c^2(x)$  is suppressed by  $\frac{g}{J} \sim \frac{1}{\log S}$ . The reason why we cannot drop it is that the density  $\varrho$  behaves as constant for large  $y$  and the integral gets large contribution of order  $\log b \sim \log S$  from large  $y$ 's (see appendix A). Since the main contribution comes from  $y \gg 1$  for  $x \sim 1$  we can neglect  $x$  in the denominator and treat  $c(x)$  as a constant! In appendix A we show that

$$c^2 = \frac{1}{\ell^2}. \tag{2.18}$$

this is how the quantity  $\ell \equiv \frac{\pi J}{\sqrt{\lambda \log S}}$  enters into our calculation.

We started from a two cut configuration whose resolvent, as is well known, is usually expressed in terms of some elliptic integrals [24]. However when  $S \rightarrow \infty$  our two branch points are effectively merging at infinity and we are therefore left with what resembles a single cut solution. This explains why we can still compute the resolvent by solving a quadratic equation.

## 2.2 Resolving quadratic equation

We see that we can treat  $c(x)$  as a constant for  $x \sim 1$ . The equation (2.13) with  $c(x) = 1/\ell$  becomes a simple quadratic equation. We can immediately solve it and find  $\mathcal{G}(x)$  for  $x \sim 1$

$$\mathcal{G}(x) = \frac{\sqrt{a^2 - x^2(1 + 4\mathcal{F})} - a}{2\ell x}, \tag{2.19}$$

where we introduced  $a$

$$a \equiv \frac{\gamma(\lambda) + J}{J} \ell = f(\lambda, \ell) + \ell. \tag{2.20}$$

It is the quantity we are aiming to compute.  $a$  by itself is related to the resolvent and its derivatives at  $x = 1$  via (2.4). Substituting (2.19) into (2.4) we will get an algebraical equation on  $a$

$$\ell = \sqrt{a^2 - \mathcal{I}^2} + \frac{8a^4\mathcal{I}^4 - 4a^2\mathcal{I}^6 + \mathcal{I}^8}{2^8 g^2 (a^2 - \mathcal{I}^2)^{7/2}} + \dots, \quad \mathcal{I} \equiv \sqrt{1 + 4\mathcal{F}(a)}, \tag{2.21}$$

where the dots are standing for some function of  $\mathcal{I}$  suppressed by  $1/g^4$ . The r.h.s. of (2.21) is some complicated function of  $a$ . We can try to solve it order by order in  $1/g$ . Since  $\mathcal{F} \sim 1/g$ , to the leading order  $\mathcal{I} \simeq 1$  and we have

$$a_0 = \sqrt{\ell^2 + 1}, \tag{2.22}$$

which is exactly the classical result (1.7). To the second order we will get

$$a_1 = \frac{2\mathcal{F}(1, a_0)}{\sqrt{\ell^2 + 1}}, \tag{2.23}$$

as we shall see that leads precisely to the correct one-loop result (1.8) of [9, 17].

For the second order iterations give

$$a_2 = -\frac{2\mathcal{F}^2(1, a_0)}{(\ell^2 + 1)^{3/2}} - \frac{8\ell^4 + 12\ell^2 + 5}{2^8 g^2 \ell^6 \sqrt{\ell^2 + 1}} + \frac{2\partial_a \mathcal{F}^2(1, a_0)}{\ell^2 + 1}. \tag{2.24}$$

In this way we can express  $a$  to an arbitrary order in  $\mathcal{F}$ .  $\mathcal{F}$  by itself is a function of  $g$ . We will denote

$$\mathcal{F}(x) = \delta\mathcal{F}(x) + \tilde{\mathcal{F}}(x) + \mathcal{O}(1/g^3) \quad , \quad \delta\mathcal{F}(x) \sim \frac{1}{g^2} \quad , \quad \tilde{\mathcal{F}}(x) \sim \frac{1}{g} \quad . \quad (2.25)$$

To compute  $\mathcal{F}(x)$  via (2.14), (2.15), (2.16) we will need to know resolvent  $\mathcal{G}(x)$ . We will see that to compute  $\mathcal{F}(x)$  it is enough to know  $\mathcal{G}(x)$  for  $x \sim 1$ , since the functions under integrals and sums in (2.14), (2.15), (2.16) are decreasing fast for large  $x$ . The resolvent can be also represented as a series in  $\mathcal{F}$  using (2.19)

$$\mathcal{G}(x) = \tilde{\mathcal{G}}(x) + \delta\mathcal{G}(x) + \mathcal{O}(\mathcal{F}^2) \quad , \quad \tilde{\mathcal{G}}(x) \equiv \frac{\sqrt{a^2 - x^2} - a}{2\ell x} \quad , \quad \delta\mathcal{G}(x) \equiv -\frac{x\mathcal{F}(x)}{\ell\sqrt{a^2 - x^2}} \quad . \quad (2.26)$$

Accordingly we also expand the density  $\varrho(x) = \tilde{\varrho}(x) + \delta\varrho(x)$

$$\tilde{\varrho}(x) = \frac{\sqrt{x-a}\sqrt{x+a}}{2\pi\ell x} \quad , \quad \delta\varrho(x) = x \frac{\mathcal{F}(x+i0) + \mathcal{F}(x-i0)}{2\pi\ell\sqrt{x-a}\sqrt{x+a}} \quad . \quad (2.27)$$

To compute  $\tilde{\mathcal{F}}$  we will use the leading term in the resolvent  $\tilde{\mathcal{G}}(x)$ , which does not depend on  $\mathcal{F}$ . Then we use  $\tilde{\mathcal{F}}$  to compute  $\mathcal{G}(x)$  with 1-loop accuracy, which is enough to compute  $\delta\mathcal{F}$ . One can continue this iterative procedure to higher orders.

In the section 3 we will compute  $\mathcal{F}$  as described above. A reader could skip the next section and continue from section 4 where the results are summarized and are used to compute  $f(\lambda, \ell)$ .

### 3. Computation of $\mathcal{F}$

#### 3.1 Hernandez-Lopez phase contribution

In this section we will calculate the contribution of the Hernandez-Lopez phase (2.15). Using (2.6) we can write

$$\begin{aligned} \mathcal{F}_{\text{HL}}(x) &= \frac{\ell^2}{g} \sum_k \frac{1}{J(x-x_k)} \frac{g\mathcal{V}_{\text{HL}}(x_k)}{J\alpha(x_k)} = \\ &= \frac{\ell^2}{g} \int_{-1}^1 \left( \frac{\mathcal{G}(x)}{x} - \frac{\mathcal{G}(x) - \mathcal{G}(1/y)}{x-1/y} + \frac{\mathcal{G}(x) - \mathcal{G}(y)}{x-y} \right) \partial_y \left( \frac{\mathcal{G}(1/y) + y^2\mathcal{G}(y) - 2y\mathcal{G}(1)}{y^2-1} \right) \frac{dy}{2\pi} \quad , \end{aligned} \quad (3.1)$$

where the path of integration goes along upper half of the unit circle  $|x|=1$ .

To calculate  $\mathcal{F}_{\text{HL}}(x)$  to the leading order in  $g$  one just replaces  $\mathcal{G}(x)$  by  $\tilde{\mathcal{G}}(x)$  from (2.26) which we denote by  $\tilde{\mathcal{F}}_{\text{HL}}(x)$ . A straightforward integration leads to<sup>2</sup>

$$\begin{aligned} \tilde{\mathcal{F}}_{\text{HL}}(x) &= -\frac{(a^2-1)}{4\pi g(x^2-1)^2} \left( 2\frac{x^2-1}{a^2-1} + 4\sqrt{\frac{a^2-x^2}{a^2-1}} \log \frac{a^2}{a^2-1} + \frac{2a^2-x^2-1}{a^2-1} \log \frac{a^4}{a^4-1} \right. \\ &\quad \left. + 2\sqrt{\frac{x^2-a^2}{a^2x^2-1}} \frac{a^2x^2+a^2-2}{a^2-1} \left[ \tan^{-1} \left( \frac{\sqrt{1-a^2x^2}}{\sqrt{a^2-1}} \right) - \tan^{-1} \left( \frac{\sqrt{1-a^2x^2}}{\sqrt{a^2-x^2}} \right) \right] \right. \\ &\quad \left. - \frac{2}{x} \left[ \frac{a^2+a^2x^2-2x^2}{a^2-1} + 2(x^2+1)\sqrt{\frac{a^2-x^2}{a^2-1}} \left[ \tanh^{-1}(x) - \tanh^{-1} \left( \frac{x\sqrt{a^2-1}}{\sqrt{a^2-x^2}} \right) \right] \right] \right) \quad . \end{aligned} \quad (3.2)$$

<sup>2</sup>One can copy (3.2) directly to *Mathematica* from table 1.



### 3.1.1 Subleading order

To the next order we need  $\mathcal{F}_{\text{HL}}(x)$  only for  $x = 1$ , according to (4.3). In this case we can simplify (3.1) further.

$$\mathcal{F}_{\text{HL}}(1) = \frac{\ell^2}{2\pi g} \int_{-1}^1 \partial_y \left( \frac{y^2 \mathcal{G}(y) + \mathcal{G}(1/y) - 2y\mathcal{G}(1)}{y^2 - 1} \right) \left( \frac{y\mathcal{G}(y) + y\mathcal{G}(1/y) - 2\mathcal{G}(1)}{y^2 - 1} \right) dy. \quad (3.3)$$

Substituting  $\mathcal{G}(x) = \tilde{\mathcal{G}}(x) + \delta\mathcal{G}(x)$  and taking the linear in  $\delta\mathcal{G}$  term, after integration by parts we find

$$\delta\mathcal{F}_{\text{HL}}(1) = \frac{\ell^2}{g} \int_{-1}^1 \left( \frac{2C(y)\delta\mathcal{G}(1)}{y} + C(1/y)\delta\mathcal{G}(1/y) - C(y)\delta\mathcal{G}(y) \right) \frac{dy}{4\pi y}, \quad (3.4)$$

where

$$C(y) = \frac{y^2}{\ell(y^2 - 1)^2} \left( 2\sqrt{a^2 - 1} - \sqrt{a^2 - y^2} - \frac{a^2 - 1}{\sqrt{a^2 - 1/y^2}} \right). \quad (3.5)$$

Changing coordinates  $y \rightarrow 1/y$  in the second term and deforming the contour to the real axe we will get the following very simple expression

$$\delta\mathcal{F}_{\text{HL}}(1, a) = \frac{\ell^2}{\pi g} \text{Re} \left[ \int_0^1 \left( \frac{\delta\mathcal{G}(1)}{y} - \delta\mathcal{G}(y) \right) \frac{C(y)dy}{y} \right]. \quad (3.6)$$

We need only  $\delta\mathcal{G}(x)$  to be computed. This will be achieved in the next section.

### 3.2 Anomalous contribution

The equation (2.16) should be understood in the following sense. We first expand formally (2.16) in powers of  $1/g$  and then perform summation over  $k$ .<sup>3</sup> To sum over  $k$  one can use that the expression which we have to sum has no poles on the cut  $\mathcal{C}$  and we can simply multiply it by the resolvent and integrate around the contour encircling only the singularities of the resolvent  $\mathcal{G}$

$$\mathcal{F}_{\text{An}}(x) = \frac{\ell^2}{J} \oint_{\mathcal{C}} \left( \frac{\pi \partial_y \tilde{\rho} [\coth(\pi \tilde{\rho}) - 1/\pi \tilde{\rho}]}{x - y} + \frac{\partial_y ([\coth(\pi \tilde{\rho}) - 1/\pi \tilde{\rho}] \pi \delta \rho)}{x - y} \right) \mathcal{G}(y) \frac{dy}{2\pi i}. \quad (3.7)$$

At the next stage we have also to expand  $\mathcal{G}$ . Each term in the expansion in  $1/g$  will have a branch cut instead of a collection of poles at positions of the Bethe roots. The sub-leading  $1/g$  term in the expansion should behave as  $-\frac{1}{4J(x-a)}$  close to the branch points as we shall see (see also [25]). This term is  $g/J$  suppressed and thus is missing in the above analysis which was done to the leading order in  $g/J$ . To see this near branch point behavior we have to go back to the equation (2.5) and rewrite it in the continuous limit as [24, 25]

$$\frac{2\pi n}{J\alpha(x)} = -2\mathcal{G} - \frac{\gamma(g) + J}{Jx} - \frac{\mathcal{V}_{\text{HL}}(x_k)}{J\alpha(x_k)} - \frac{\pi\rho'(x) \coth(\pi\rho)}{J} + \mathcal{O}(1/g^2), \quad (3.8)$$

---

<sup>3</sup>This simple prescription was worked out based on the Airy function behavior of the resolvent close to the branch points [25] in collaboration with Andrzej Jarosz. This prescription was derived for  $sl(2)$  Heisenberg spin chain only. Here we are assuming that it is still valid for the all-loop  $sl(2)$  Bethe ansatz. That could be done since the near branch point behavior is very universal.

where

$$\mathcal{G}(x) \equiv \frac{\mathcal{G}(x+i0) + \mathcal{G}(x-i0)}{2}. \quad (3.9)$$

Close to a branch point density goes to zero as a square root  $\rho \sim \sqrt{x-a}$ . The last term becomes singular and we have

$$\mathcal{G}(x) \simeq -\frac{\pi\rho' \coth(\pi\rho)}{2J} \simeq -\frac{1}{4J(x-a)} \quad (3.10)$$

which proves our claim. For more details about behavior of resolvents near branch points we refer to [25].

Although this singularity in  $\mathcal{G}$  is suppressed by  $g/J$  it will lead to a finite contribution which we call “boundary term”.

### 3.2.1 Boundary term

We replace  $\mathcal{G}$  in (3.7) by  $-\frac{1}{4(y-a)} - \frac{1}{4(y+a)}$ . The contour of integration now contains only 2 poles inside and we just have to evaluate the expression in the brackets at  $y = \pm a$ . Consider first the contribution from  $y = a$ .

$$-\ell^2 \frac{\pi \partial_y \rho [\coth(\pi\rho) - 1/\pi\rho]}{4J^2(x-a)} \simeq -\frac{\ell^2}{4J^2(x-a)} \partial_x \frac{\pi^2 \rho^2}{6} = -\frac{a^3}{48g^2(a^2-1)^2(x-a)}. \quad (3.11)$$

Taking into account a similar contribution from  $x = -a$  we will get

$$\mathcal{F}_{\text{An}}^{\text{boundary}}(x) = \frac{a^4}{24g^2(a^2-1)^2(a^2-x^2)}. \quad (3.12)$$

We see that all factors of  $J$  cancel and we get a finite contribution. For  $x = 1$  and  $a = a_0 = \sqrt{\ell^2 + 1}$  we get

$$\mathcal{F}_{\text{An}}^{\text{boundary}}(1, a_0) = \frac{(\ell^2 + 1)^2}{24g^2\ell^6}. \quad (3.13)$$

We see that this term is very singular in the limit  $\ell \rightarrow 0$ . However then we add all pieces together the full result is completely finite as we shall see.

### 3.2.2 Bulk contribution

In this section we will drop poles of the resolvent at the branch points. This implies that we can pass to the integration along the cut with density  $\varrho(x)$

$$\mathcal{F}_{\text{An}}^{\text{bulk}}(x) \equiv \ell^2 \int_{\mathcal{C}} \left( \frac{\pi \partial_y \tilde{\rho} [\coth(\pi\tilde{\rho}) - 1/\pi\tilde{\rho}]}{x-y} + \frac{\partial_y ([\coth(\pi\tilde{\rho}) - 1/\pi\tilde{\rho}] \pi \delta\rho)}{x-y} \right) \frac{\varrho(y) dy}{J}. \quad (3.14)$$

Where we use notations introduced above

$$\tilde{\rho}(x) = J\alpha(x)\tilde{\varrho}(x), \quad \delta\rho(x) = J\alpha(x)\delta\varrho(x), \quad \alpha(x) = \frac{x^2}{g(x^2-1)}. \quad (3.15)$$

In (3.14) there are contributions of both  $1/g$  and  $1/g^2$  orders. We split  $\mathcal{F}_{\text{An}}^{\text{bulk}}(x)$  further into  $\tilde{\mathcal{F}}_{\text{An}}(x)$  and  $\delta\mathcal{F}_{\text{An}}^{\text{bulk}}(x)$  as defined below.

$$\tilde{\mathcal{F}}_{\text{An}}(x) \equiv \ell^2 \int_{\mathcal{C}} \frac{\pi \tilde{\varrho} \partial_y \tilde{\rho} [\coth(\pi\tilde{\rho}) - 1/\pi\tilde{\rho}]}{x-y} \frac{dy}{J} = \ell^2 \int_{\mathcal{C}} \frac{\pi \tilde{\varrho} \partial_y \tilde{\rho}}{x-y} \frac{dy}{J}, \quad (3.16)$$

where in the last equality we use that from (3.15)  $\tilde{\rho}(y) \sim J/g \gg 1$  which allowed us to replace  $[\coth(\pi\tilde{\rho}) - 1/\pi\tilde{\rho}]$  by 1 in the second equality. Using (2.27) one can easily evaluate the integral (3.16) to get<sup>4</sup>

$$\tilde{\mathcal{F}}_{\text{An}}(x) = \frac{x \log \frac{a-1}{a+1} (1+x^2-2a^2) + 2ax(x^2-1) + \log \frac{a-x}{a+x} (a^2x^2+a^2-2x^2)}{4\pi g x(x^2-1)^2}. \quad (3.17)$$

### 3.2.3 Second order

The last contribution of  $1/g^2$  order into  $\mathcal{F}_{\text{An}}(x)$  reads

$$\begin{aligned} \delta\mathcal{F}_{\text{An}}^{\text{bulk}}(x) &\equiv \ell^2 \int_{\mathcal{C}} \left( \frac{\partial_y \tilde{\rho} [\coth(\pi\tilde{\rho}) - 1/\pi\tilde{\rho}] \pi \delta \varrho}{x-y} + \frac{\tilde{\rho} \partial_y ([\coth(\pi\tilde{\rho}) - 1/\pi\tilde{\rho}] \pi \delta \rho)}{x-y} \right) \frac{dy}{J} \\ &= \ell^2 \int_{\mathcal{C}} \frac{\partial_y (\pi \delta \rho \tilde{\rho} [\coth(\pi\tilde{\rho}) - 1/\pi\tilde{\rho}])}{x-y} \frac{dy}{J^2 \alpha(y)} \\ &= - \int_{\mathcal{C}} \partial_y \left( \frac{1}{\alpha(y)(x-y)} \right) \frac{\pi \ell^2 \delta \rho \tilde{\rho} dy}{J^2}. \end{aligned} \quad (3.18)$$

To evaluate this integral we need  $\delta\rho$  which can be expressed in terms of  $\mathcal{F}$  (2.27). We have

$$\frac{\pi \ell^2 \delta \rho \tilde{\rho}}{J^2} = y^4 \frac{\tilde{\mathcal{F}}(y+i0) + \tilde{\mathcal{F}}(y-i0)}{4\pi g^2 (y^2-1)^2}. \quad (3.19)$$

Setting  $x=1$  we will get the following simple result

$$\delta\mathcal{F}_{\text{An}}^{\text{bulk}}(1, a) = -\frac{1}{g} \int_{\mathcal{C}} \frac{\tilde{\mathcal{F}}(y+i0) + \tilde{\mathcal{F}}(y-i0)}{4\pi} \frac{y^2 dy}{(y^2-1)^2}, \quad (3.20)$$

where  $\tilde{\mathcal{F}} = \tilde{\mathcal{F}}_{\text{HL}} + \tilde{\mathcal{F}}_{\text{An}}$ . Using (3.2) and (3.17) one can see that

$$\begin{aligned} \tilde{\mathcal{F}}(x+i0) + \tilde{\mathcal{F}}(x-i0) &= \frac{1}{\pi g (x^2-1)^2} \left( (a-1)(x^2-1) + \frac{1+x^2-2a^2}{2} \log \frac{(a-1)a^4}{(a+1)(a^4-1)} \right. \\ &+ \frac{a^2x^2+a^2-2x^2}{2x} \log \frac{(x+1)(x-a)}{(x-1)(x+a)} + (2-a^2-a^2x^2) \sqrt{\frac{x^2-a^2}{a^2x^2-1}} \arctan \sqrt{\frac{x^2-a^2}{a^2x^2-1}} \\ &\left. + 2 \frac{x^2+1}{x} \sqrt{(a^2-1)(x^2-a^2)} \arctan \sqrt{\frac{x^2-a^2}{x^2(a^2-1)}} \right). \end{aligned} \quad (3.21)$$

The integral (3.20) can be computed numerically for an arbitrary value of  $a^5$  or expanded in powers of  $\ell$ . The result of this expansion is give in eq.(B.1).

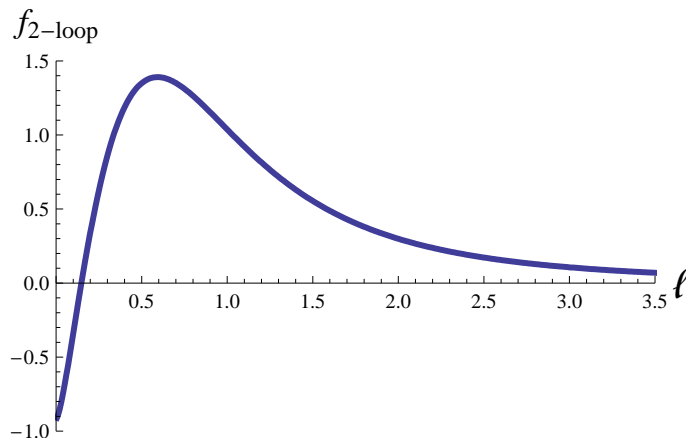
### 3.3 Computation of $\mathcal{F}_0$

The only piece left to compute is  $\mathcal{F}_0$  (2.14). Since it is already suppressed by  $1/g^2$  this contribution is especially simple to compute. We immediately evaluate integration using (2.27)

$$\mathcal{F}_0(1) = -\frac{24a^4 + 32a^2 - 7}{2^9 3 (a^2-1)^3 g^2} = -\frac{24\ell^4 + 80\ell^2 + 49}{2^9 3 g^2 \ell^6} \quad (3.22)$$

<sup>4</sup>One can copy (3.17) directly to *Mathematica* from table 1.

<sup>5</sup>In table 2 we give a *Mathematica* code which computes this integral numerically.



**Figure 1:** Two-loop correction to the generalized cusp anomalous dimension as a function of  $\ell = \frac{\pi J}{\sqrt{\lambda} \log S}$ . It interpolates between minus Catalan’s constant  $-C \simeq -0.916$  at small  $\ell$ ’s and 0 at large  $\ell$ ’s.

#### 4. Scaling function at one and two loops

Using expressions for  $a_1$  and  $a_2$  in terms of  $\mathcal{F}$  (2.23), (2.23) and results of the previous section, where  $\mathcal{F}$  was computed up to  $1/g^2$  order we will compute the generalized scaling function  $f(g, \ell)$  with the two-loop accuracy in this section.

##### 4.1 One-loop order

Having  $\tilde{\mathcal{F}} = \tilde{\mathcal{F}}_{\text{HL}} + \tilde{\mathcal{F}}_{\text{An}}$  computed we can immediately compute the one-loop energy density using (2.23)

$$f_{1\text{-loop}}(\ell) = 8\pi g \frac{\tilde{\mathcal{F}}_{\text{HL}}(1) + \tilde{\mathcal{F}}_{\text{An}}(1)}{\sqrt{\ell^2 + 1}} \Big|_{a=\sqrt{\ell^2+1}}. \quad (4.1)$$

From (3.2), (3.17) we have for  $x = 1$

$$\tilde{\mathcal{F}}_{\text{HL}} + \tilde{\mathcal{F}}_{\text{An}} = \frac{2(a-1) + 4a^2 \log \frac{a^2}{a+1} + \log \frac{(a-1)^2}{a^2+1} - a^2 \log(a-1)^2(a^2+1)}{4\sqrt{\lambda}}, \quad (4.2)$$

and we precisely reproduce (1.8) by setting  $a = a_0 = \sqrt{\ell^2 + 1}$ !

##### 4.2 Two-loop order

Now we can write down our 2-loop result. From (2.20), (2.23) and (2.24) we have

$$f_{2\text{-loop}} = \frac{16\pi^2}{\sqrt{\ell^2 + 1}} \left( \frac{2g^2 \partial_a \tilde{\mathcal{F}}^2(a_0)}{\sqrt{\ell^2 + 1}} - \frac{2g^2 \tilde{\mathcal{F}}^2(a_0)}{\ell^2 + 1} + 2g^2 \delta\mathcal{F} - \left( \frac{5}{256\ell^6} + \frac{3}{64\ell^4} + \frac{1}{32\ell^2} \right) \right), \quad (4.3)$$

where  $a_0 = \sqrt{\ell^2 + 1}$  and

$$\tilde{\mathcal{F}} = \tilde{\mathcal{F}}_{\text{HL}} + \tilde{\mathcal{F}}_{\text{An}} \quad (4.4)$$

$$\delta\mathcal{F} = \mathcal{F}_0 + \delta\mathcal{F}_{\text{HL}} + \delta\mathcal{F}_{\text{An}}^{\text{bulk}} + \delta\mathcal{F}_{\text{An}}^{\text{boundary}}. \quad (4.5)$$

The quantities in the r.h.s. of the first line are given by (3.2), (3.17) and of the second line by (3.22), (3.6), (3.13), (3.20).  $\delta\mathcal{F}_{\text{HL}}$  and  $\delta\mathcal{F}_{\text{HL}}^{\text{boundary}}$  could be represented explicitly as single integrals. To evaluate them numerically one can use the *Mathematica* code form table 3. In appendix B we give an expansion of these integrals in power series in  $\ell$  up to  $\ell^6$  order.

Let us see that the result (4.3) is finite in the small  $\ell$  limit. This will be already a very nontrivial test of our calculation because a priori the r.h.s. is divergent as  $1/\ell^6$ . For the expansion in  $\ell$  we have

$$\frac{2g^2\partial_a\tilde{\mathcal{F}}^2}{\sqrt{\ell^2+1}} \simeq \frac{\log 8 \log \ell}{4\pi^2} + \frac{\log^2 8 - \log 8}{16\pi^2} \quad (4.6)$$

$$-\frac{2g^2\tilde{\mathcal{F}}^2}{\ell^2+1} \simeq -\frac{\log^2 8}{32\pi^2} \quad (4.7)$$

$$2g^2\mathcal{F}_0 = -\frac{49}{768\ell^6} - \frac{5}{48\ell^4} - \frac{1}{32\ell^2} \quad (4.8)$$

$$2g^2\delta\mathcal{F}_2^{\text{boundary}} = \frac{1}{12\ell^6} + \frac{1}{6\ell^4} + \frac{1}{12\ell^2} \quad (4.9)$$

Using expansion from appendix B we have

$$2g^2\delta\mathcal{F}_{\text{HL}} \simeq \frac{1}{\ell^4} \left( -\frac{1}{4\pi^2} + \frac{1}{24\pi} + \frac{\log 8}{24\pi^2} \right) + \frac{1}{\ell^2} \left( -\frac{1}{48} - \frac{1}{2\pi^2} + \frac{1}{8\pi} \right) \quad (4.10)$$

$$+ \left( -\frac{\log \ell}{32\pi} - \frac{5 \log 8 \log \ell}{16\pi^2} - \frac{\log^2 8}{96\pi^2} - \frac{\log 8}{64\pi^2} + \frac{\log 8}{64\pi} + \frac{9}{128\pi} - \frac{C}{8\pi^2} - \frac{1}{16\pi^2} - \frac{1}{64} \right)$$

$$2g^2\delta\mathcal{F}_{\text{An}}^{\text{bulk}} \simeq \frac{1}{\ell^4} \left( -\frac{1}{64} + \frac{1}{4\pi^2} - \frac{1}{24\pi} - \frac{\log 8}{24\pi^2} \right) + \frac{1}{\ell^2} \left( \frac{1}{2\pi^2} - \frac{1}{8\pi} \right) \quad (4.11)$$

$$+ \left( \frac{\log \ell}{32\pi} + \frac{\log 8 \log \ell}{16\pi^2} - \frac{\log^2 8}{48\pi^2} - \frac{\log 8}{64\pi} + \frac{5 \log 8}{64\pi^2} - \frac{9}{128\pi} + \frac{C}{16\pi^2} + \frac{1}{16\pi^2} + \frac{1}{64} \right)$$

Where  $C \simeq 0.916$  is Catalan's constant. We see that indeed all divergent terms cancel and only the terms with Catalan's constant survive leading to  $f_{2\text{-loop}} = -C + \mathcal{O}(\ell^2)$  in complete agreement with [20]! Note that only 2 out of 44 terms survive when we sum all up! This huge cancelation entangles nontrivially all the six contributions of a very different nature. In (6.2) we expanded  $f_{2\text{-loop}}(\ell)$  further in  $\ell$ .

## 5. Leading logarithms

As one can see the point  $\ell = 0$  is a singular point of the function  $f_{1\text{-loop}}$  (1.8). The singular part is

$$f_{1\text{-loop}}(\ell) = -\frac{\ell^2 \log \ell^2}{\sqrt{\ell^2+1}}. \quad (5.1)$$

It contains  $\log \ell$  singularity. At two loops as one can see from (6.2) there is also  $\log^2 \ell$  singularity. In this section we are aiming to understand how these singularities appear in our calculation. The central object in our calculation is  $\mathcal{F}(\lambda, a)$ . One can see from (4.2) that with 1-loop precision, up to regular at  $a = 1$  terms

$$\mathcal{F} \sim -\frac{(a^2-1)}{2\sqrt{\lambda}} \log(a-1) + \mathcal{O}\left(\frac{\log(a-1)}{\lambda}\right). \quad (5.2)$$

2-loops correction in  $\mathcal{F}$  also contains only  $\log(a - 1)$  to the first power as one can see from expansion in appendix B. This observation allows us to assume that n-loop correction will contain  $\log^{n-1}(a - 1)$  at most. Let us use this assumption about  $\mathcal{F}$  to compute the  $\log \ell$  terms to the maximal power at each order in  $1/\sqrt{\lambda}$ . We can use (2.21) and drop terms in r.h.s. suppressed by  $1/\lambda$ , since they cannot contain log terms to the maximal power. Concerning the leading log terms the equation

$$a = \sqrt{1 + \ell^2 + 4\mathcal{F}(a)} \tag{5.3}$$

is exact. For  $\mathcal{F}$  it is enough to take 1-loop expression (5.2) as far as the leading logarithms are considered. We will get some simple quadratic equation on  $a$  which leads to

$$a_{LL} = \sqrt{1 + \frac{\ell^2}{1 + 2\frac{\log(a-1)}{\sqrt{\lambda}}}} \tag{5.4}$$

Using that  $f = a_{LL} - \ell$  and expanding the above equation one finds

$$f_{LL} = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{k_{nm} \ell^{2m} \log^n \ell}{\lambda^{n/2}}, \quad k_{nm} = (-1)^{n+m+1} \frac{4^n (2m-3)!! (m+n-1)!}{2^m m! n! (m-1)!} \tag{5.5}$$

In particular  $k_{n1} = (-1)^n 4^n / 2$  in agreement with [19]. The terms with  $m > 1$  could not be captured by the  $O(6)$  sigma model. However they could correspond to a marginal operators with many derivatives which should be added to the effective  $O(6)$  sigma model action considered by [19].

## 6. Conclusions

In this paper we consider the  $sl(2)$  sector of the AdS/CFT correspondence. We calculate the energy of the string rotating in  $AdS_3 \times S^1$  with angular momenta  $S$  and  $J$  correspondingly. In the limit  $S, J \rightarrow \infty$  with  $\ell = \frac{J\pi}{\sqrt{\lambda} \log S}$  fixed we compute the 2-loop correction to its energy.

From the gauge side of the duality this corresponds to operators of the form  $\text{Tr}(D^S \Phi^J)$  with twist  $J$ . In this limit the anomalous dimensions of the operators scale like  $J$  and one defines the generalized scaling function  $f(\lambda, \ell) = \gamma(\lambda)\ell/J$ . The strong coupling expansion of the generalized scaling function is organized in the negative half-integer powers of  $\lambda$

$$f(\lambda, \ell) = f_{cl}(\ell) + \lambda^{-1/2} f_{1\text{-loop}}(1)(\ell) + \lambda^{-1} f_{2\text{-loop}}(\ell) + \dots, \tag{6.1}$$

where the first term is the classical energy-density of the string. The second term was computed in [9, 17]. The last term is computed in this paper as a function of  $\ell$ . Its small  $\ell$  expansion reads

$$\begin{aligned} f_{2\text{-loop}} = & -C + \ell^2 \left( 8 \log^2 \ell - 6 \log \ell - \frac{\log 8}{2} + \frac{11}{4} \right) \\ & + \ell^4 \left( -6 \log^2 \ell - \frac{7 \log \ell}{6} + \log 8 \log \ell - \frac{\log^2 8}{8} + \frac{11 \log 8}{24} - \frac{233}{576} + \frac{3C}{32} \right) \\ & + \ell^6 \left( 6 \log^2 \ell - \frac{26 \log \ell}{15} - \frac{3 \log 8 \log \ell}{2} + \frac{3 \log^2 8}{16} - \frac{17 \log 8}{30} + \frac{12779}{14400} - \frac{3C}{32} \right) + \dots \end{aligned} \tag{6.2}$$

The leading term agrees with [20]. Also the  $\ell^2 \log^2 \ell$  and  $\ell^2 \log \ell$  terms agree with [19] and [22]. However the  $\ell^2$  coefficient does not match earlier results of [22]. It is important to understand this mismatch and to reproduce the higher terms in  $\ell^2$  directly from the string sigma model Feynman diagrams. That will provide very a nontrivial test of the two-loop coefficient in the dressing phase and integrability of the  $AdS_5 \times S^5$  super-string sigma model.

In this paper we also compute at each order in  $1/\sqrt{\lambda}$  all the terms containing  $\log l$  to the maximal power (5.5)

$$\begin{aligned}
 f(\lambda, \ell) \sim & \frac{\log \ell}{\lambda^{1/2}} (-2\ell^2 + \ell^4 - 3/4\ell^6 + \dots) \\
 & + \frac{\log^2 \ell}{\lambda} (8\ell^2 - 6\ell^4 + 6\ell^6 + \dots) \\
 & + \frac{\log^3 \ell}{\lambda^{3/2}} (-32\ell^2 + 32\ell^4 - 40\ell^6 + \dots) + \dots
 \end{aligned}
 \tag{6.3}$$

The  $\ell^2$  terms reproduce earlier predictions by Alday and Maldacena [19]. We have, however, a disagreement with [22] for what concerns the  $1/\lambda$  terms.

We show that these logarithmic terms (6.3) are only probing the Hernandez-Lopez dressing phase and are not sensitive to the higher terms in the expansion in  $1/\sqrt{\lambda}$  of the dressing phase. We also argue that the sub-leading logarithms could be computed using our method. They should be sensitive only to first few terms in the strong coupling expansion of the dressing phase.

As future work, it could be interesting to compute the large  $\ell$  expansion of the scaling function. The calculation should simplify and several worldsheet loops could be doable. It would also be interesting to compute all  $\log \ell$  terms in the sub-leading power at each order of perturbation theory and possibly check our results numerically.

**Note added.** Interesting papers [30–32] appeared while this paper was in preparation during the last two days. Some of results seems to be similar. All these papers are based on a different approach.

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## A. Calculation of $c(x)$

In this appendix we will calculate the function  $c(x)$  defined in (2.17) as

$$c^2(x) = -16\pi \frac{g}{J} \int_a^b \frac{\varrho(y)}{x-y} \left(1 - \frac{1}{y^2}\right) dy, \tag{A.1}$$

due to the suppression by  $1/J$  the only chance to get a finite result is to assume that the density for  $1 \ll y \ll b$  goes to a constant. Then from large  $y$ 's we will get a big contribution of order  $\log b \sim \log S \sim J$ . We see that to compute  $c^2$  we only need some information about  $\varrho(y)$  when  $y$  is large. In particular for  $x \sim 1$  we simply have

$$c^2(x) \simeq 16\pi \frac{g}{J} \int_a^b \frac{\varrho(y)}{y} dy . \quad (\text{A.2})$$

To find behavior of  $\varrho(x)$  for large  $x$  we can still use (2.13). For  $1 \ll x \ll b \sim \frac{S}{\sqrt{\lambda}}$  it reads

$$-\frac{c^2(x)}{4} = \mathcal{G}^2(x) + \mathcal{O}(1/x) . \quad (\text{A.3})$$

From (2.27) we see that for large  $y$  the density behaves as a constant  $\varrho(y) \simeq \beta$ . Let us try to plug this into (A.3). What we will get is

$$-\frac{c^2(x)}{4} = -\frac{\pi\beta}{\ell \log S} \log(S/x) = -\frac{\pi\beta}{\ell} \left(1 - \frac{\log x}{\log S}\right) . \quad (\text{A.4})$$

Whereas in the r.h.s. of (A.3) we get  $\mathcal{G}^2 \simeq (\pi i \varrho)^2 \simeq -\pi^2 \beta^2$  and we see that (A.3) cannot be satisfied at large  $x$  when  $\log x \sim \log S$ . This simply means that  $\varrho(x)$  is not a constant but it could also contains terms  $\frac{\log^n x}{\log^n S}$  which are not relevant when  $x$  becomes smaller. This terms are not visible in (2.27). In fact one can see that the only consistent with (A.3) combination of  $\frac{\log^n x}{\log^n S}$  is

$$\varrho \simeq \beta_1 + \beta_2 \frac{\log x}{\log S} , \quad 1 \ll x \ll S , \quad (\text{A.5})$$

integrating with this density we will get

$$-\frac{c^2(x)}{4} \simeq -\frac{\pi}{\ell} \left[ \beta_1 \left(1 - \frac{\log x}{\log S}\right) + \frac{\beta_2}{2} \left(1 - \frac{\log^2 x}{\log^2 S}\right) \right] . \quad (\text{A.6})$$

We have to equate this with

$$\mathcal{G}^2(x) \simeq -\pi^2 \left( \beta_1^2 + 2\beta_1\beta_2 \frac{\log x}{\log S} + \beta_2^2 \frac{\log^2 x}{\log^2 S} \right) . \quad (\text{A.7})$$

Note that we get three equations on two unknowns  $\beta_1$  and  $\beta_2$ . All of them can be resolved at the same time by setting

$$\beta_1 = \frac{1}{2\pi\ell} , \quad \beta_2 = -\frac{1}{2\pi\ell} , \quad (\text{A.8})$$

so that

$$c^2(x) = \frac{1}{\ell^2} \frac{\log^2(S/x)}{\log^2(S)} , \quad (\text{A.9})$$

in particular when  $x \sim 1$  we get (2.18). The  $\log x$  terms in the density are suppressed by  $1/\log S$  when  $x \sim 1$  and of course does not contradict to (2.27) where this terms are absent.

Note that for the density we finally get

$$\rho \simeq \frac{J}{g} \varrho \simeq \frac{J}{2\pi g \ell} \left(1 - \frac{\log x}{\log S}\right) = \frac{2}{\pi} \log(S/x) , \quad 1 \ll x \ll S , \quad (\text{A.10})$$



```

tF1[x_] = -(((a^2-1)/(4 g Pi (x^2 - 1)^2)) ((2 (x^2 - 1))/(a^2 - 1) + 4 Sqrt[(a^2 - x^2)/(a^2 - 1)] Log[a^2/(a^2 - 1)] + ((2 a^2 - x^2 - 1) Log[a^4/(a^4 - 1)])/(a^2 - 1) + (((2 x^2 - a^2 (x^2 + 1))/(a^2 - 1) - 2 Sqrt[(a^2 - x^2)/(a^2 - 1)] (x^2 + 1)) Log[((x + 1) (Sqrt[a^2 - 1] x - Sqrt[a^2 - x^2]))/(x - 1) (Sqrt[a^2 - 1] x + Sqrt[a^2 - x^2]))])/ x - (I Sqrt[a^2 - x^2] (a^2 (x^2 + 1) - 2) Log[-((Sqrt[a^2 - 1] + I Sqrt[1 - a^2 x^2]) (I Sqrt[a^2 - x^2] + Sqrt[1 - a^2 x^2]))/(Sqrt[a^2 - 1] - I Sqrt[1 - a^2 x^2]) (Sqrt[1 - a^2 x^2] - I Sqrt[a^2 - x^2]))])/((a^2 - 1) Sqrt[1 - a^2 x^2]))

tF2[x_] = (2 a x (x^2 - 1) + (x^3 - 2 a^2 x + x) Log[(a - 1)/(a + 1)] + (a^2 (x^2 + 1) - 2 x^2) Log[(a - x)/(a + x)])/(4 g Pi x (x^2 - 1)^2)

```

**Table 1:** Expressions for  $\tilde{\mathcal{F}}_{\text{HL}}(x)$  and  $\tilde{\mathcal{F}}_{\text{An}}(x)$  from (3.2) and (3.17)

```

stF[x_] = (1/(g Pi (x^2 - 1)^2)) ((a - 1) (x^2 - 1) + Sqrt[(x^2 - a^2)/(a^2 x^2 - 1)] (2 - a^2 (x^2 + 1)) ArcTan[Sqrt[(x^2 - a^2)/(a^2 x^2 - 1)]] + 2 (x^2 + 1) Sqrt[((a^2 - 1) (x^2 - a^2))/x^2] ArcTan[Sqrt[x^2 - a^2]/Sqrt[a^2 x^2 - x^2]] + (1/2) (-2 a^2 + x^2 + 1) Log[(a - 1)/(a + 1)] + (1/2) (-2 a^2 + x^2 + 1) Log[a^4/(a^4 - 1)] + ((a^2 (x^2 + 1))/(2 x) - x) Log[(x + 1)/(x - 1)] + ((a^2 (x^2 + 1))/(2 x) - x) Log[(x - a)/(a + x)])

```

**Table 2:** Expression for  $\tilde{\mathcal{F}}(x + i0) + \tilde{\mathcal{F}}(x - i0)$  from (3.21)

```

Off[Series::ztest, NIntegrate::slwcon];
dF2bulk[1, a0_] := -(2/g^2) NIntegrate[Re[(g stF[y] y^2)/(4 Pi (y^2 - 1)^2) /. a -> a0], {y, a0, Infinity}, WorkingPrecision -> 20, MaxRecursion -> 40]

c[y_] := (y^2/(1 (y^2-1)^2)) (2 Sqrt[a^2-1]-Sqrt[a^2-y^2]-(a^2-1)/Sqrt[a^2-1/y^2]);
dG[x_] = -((x (tF1[x] + tF2[x]))/(1 Sqrt[a^2 - x^2]));
dG[1] = Normal[Simplify[Series[dG[x] /. a -> Zeta[3], {x, 1, 0}] /. Zeta[3] -> a]];
dF1[1, a0_] := (1/(Pi g^2)) Re[NIntegrate[g 1^2 c[y] (dG[1]/y - dG[y]) (1/y) /. a -> a0, {y, 1, 1 - 10^(-30)}, WorkingPrecision -> 30, MaxRecursion -> 40]]

dF[1, 1_] := (40 1^4 + 48 1^2 + 15)/(1536 g^2 1^6) + dF1[1, Sqrt[1^2 + 1]] + dF2bulk[1, Sqrt[1^2 + 1]]

tF[1] = Normal[Simplify[Series[tF1[x] + tF2[x] /. a -> Zeta[3], {x, 1, 0}]] /. Zeta[3] -> a];
f2loop[1_] := ((16 Pi^2 g^2)/Sqrt[1^2 + 1]) ((2 D[tF[1]^2, a])/Sqrt[1^2 + 1] - (2 tF[1]^2)/(1^2 + 1) - (1/g^2) (5/(256 1^6) + 3/(64 1^4) + 1/(32 1^2)) + 2 dF[1, 1]) /. a -> Sqrt[1^2 + 1];

```

**Table 3:** Numerical evaluation of  $\delta\mathcal{F}(a_0)$  from (4.5) and  $f_{2\text{-loop}}(\ell)$  from (4.3)

which is exactly what one gets from the well-known Korchemsky's density [29]

$$\rho_0(u) = \frac{1}{\pi} \log \frac{1 + \sqrt{1 - 4u^2/S^2}}{1 - \sqrt{1 - 4u^2/S^2}} \simeq \frac{2}{\pi} \log(S/u) \quad , \quad |u| \ll S, \quad (\text{A.11})$$

that can be used as an alternative derivation.<sup>6</sup>

## B. Expansion in $\ell$

In section 4 we expressed the 2-loop result for the generalized scaling function  $f(\lambda)$  in terms

<sup>6</sup>We would like to thank D.Serban for pointing that out.

of two single integrals (3.6) and (3.20) of a rather complicated functions. In this appendix we give results of the expansion of these integrals in powers of  $\ell$ .

Expansion of (3.20) reads

$$\begin{aligned}
 g^2 \delta \mathcal{F}_{\text{An}}^{\text{bulk}} \simeq & \frac{1}{\ell^4} \left( -\frac{1}{128} + \frac{1}{8\pi^2} - \frac{1}{48\pi} - \frac{\log 8}{48\pi^2} \right) + \frac{1}{\ell^2} \left( \frac{1}{4\pi^2} - \frac{1}{16\pi} \right) \\
 & + \ell^0 \left( \left( \frac{1}{64\pi} + \frac{\log 8}{32\pi^2} \right) \log \ell - \frac{\log^2 8}{96\pi^2} - \frac{\log 8}{128\pi} + \frac{5 \log 8}{128\pi^2} - \frac{9}{256\pi} + \frac{C}{32\pi^2} + \frac{1}{32\pi^2} + \frac{1}{128} \right) \\
 & + \ell^2 \left( \left( -\frac{3}{64\pi^2} + \frac{\log 8}{32\pi^2} \right) \log \ell - \frac{\log^2 8}{96\pi^2} + \frac{\log 8}{48\pi^2} - \frac{5}{768\pi} - \frac{17}{384\pi^2} \right) \\
 & + \ell^4 \left( \left( -\frac{43}{3072\pi^2} - \frac{3}{2048\pi} \right) \log \ell - \frac{49 \log 8}{18432\pi^2} + \frac{3 \log 8}{4096\pi} + \frac{15}{16384\pi} - \frac{3C}{1024\pi^2} + \frac{1753}{73728\pi^2} \right) \\
 & + \ell^6 \left( \left( \frac{113}{15360\pi^2} + \frac{1}{2048\pi} \right) \log \ell - \frac{\log 8}{5760\pi^2} - \frac{\log 8}{4096\pi} - \frac{1}{49152\pi} + \frac{C}{1024\pi^2} - \frac{439}{76800\pi^2} \right).
 \end{aligned} \tag{B.1}$$

We computed these coefficients analytically by a rather length procedure, which we do not describe here. We checked this expansion by a numerical fit of the integral. We found that the numerical mismatch of all these coefficients is less then  $10^{-45}$ .

For the expansion of (3.6) we found

$$\begin{aligned}
 g^2 \delta \mathcal{F}_{\text{HL}} \simeq & \frac{1}{\ell^4} \left( -\frac{1}{8\pi^2} + \frac{1}{48\pi} + \frac{\log 8}{48\pi^2} \right) + \frac{1}{\ell^2} \left( -\frac{1}{96} - \frac{1}{4\pi^2} + \frac{1}{16\pi} \right) \\
 & + \ell^0 \left( \left( -\frac{1}{64\pi} - \frac{5 \log 8}{32\pi^2} \right) \log \ell - \frac{\log^2 8}{192\pi^2} - \frac{\log 8}{128\pi^2} + \frac{\log 8}{128\pi} + \frac{9}{256\pi} - \frac{C}{16\pi^2} - \frac{1}{32\pi^2} - \frac{1}{128} \right) \\
 & + \ell^2 \left( \left( \frac{7}{64\pi^2} - \frac{3 \log 8}{32\pi^2} \right) \log \ell - \frac{\log^2 8}{192\pi^2} + \frac{25 \log 8}{384\pi^2} + \frac{5}{768\pi} - \frac{C}{64\pi^2} + \frac{1}{12\pi^2} \right) \\
 & + \ell^4 \left( \left( \frac{49}{1024\pi^2} + \frac{3}{2048\pi} \right) \log \ell + \frac{493 \log 8}{18432\pi^2} - \frac{3 \log 8}{4096\pi} - \frac{15}{16384\pi} + \frac{5C}{512\pi^2} - \frac{2671}{24576\pi^2} \right) \\
 & + \ell^6 \left( \left( -\frac{421}{15360\pi^2} - \frac{1}{2048\pi} \right) \log \ell - \frac{1001 \log 8}{92160\pi^2} + \frac{\log 8}{4096\pi} + \frac{1}{49152\pi} - \frac{9C}{2048\pi^2} + \frac{32951}{921600\pi^2} \right).
 \end{aligned} \tag{B.2}$$

These coefficients are also checked numerically with 30 digits accuracy.

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